YESHIVA UNIVERSITY
GRADUATE PROGRAM IN MATHEMATICAL SCIENCES
TOPICS FOR THE PHD QUALIFYING EXAMINATION

The qualifying examination in mathematical sciences covers three areas:

(I) Real Analysis

(II) Complex Analysis

(III) Research Area

For the first two areas, a list of topics is below. Also, a list of sample exercises for the two areas is provided. The actual exercises asked on the exam will be different from the sample exercises; being able to solve the sample exercises is not sufficient for the exam preparation.

The third exam area pertains to the research subject that the student intends to undertake, and should be established between the student and the student's adviser. The student shall request and be provided with a list of topics on the research area and a bibliography from the faculty adviser, in order to prepare for this part of the qualifying examination.

The examination is oral and is usually administered by the candidate's advisory committee. The examination on the first two areas is usually administered together, and the examination on the third area is in usually administered separately. The possible grades on the examination are Pass and Fail. The qualifying examination may be retaken once. A second failure will result in dismissal from the program.

For the qualifying exam, the student is expected to demonstrate a thorough understanding of the topics, including stating and proving theorems, defining all notions involved in those theorems, providing examples and counterexamples, and solving exercises that apply the theory.

Real Analysis:

1. Bolzano-Weierstrass Theorem. $S \subseteq \mathbb{R}^n$ is closed and bounded if and only if every sequence in S has a convergent subsequence whose limit is in S.
2. Heine-Borel Theorem. $S \subseteq \mathbb{R}^n$ is closed and bounded if and only if every open covering of S has a finite subcover.
3. Every continuous function on a closed bounded interval has an absolute maximum and an absolute minimum on that interval.
4. Every continuous function on an interval satisfies the intermediate value property.
5. The Mean Value Theorem.
6. The Fundamental Theorem of Calculus.

7. Riemann-Lebesgue Theorem (outline). A function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if it is bounded and its set of discontinuity points is of measure zero.

8. Sequences of functions (uniform convergence, properties, equi-continuity for a family of functions, Ascoli-Arzelà's theorem).

Complex Analysis:

1. If f is complex differentiable at z then the Cauchy-Riemann equations are satisfied at z.

2. If the partial derivatives of u and v exist and are continuous at (x, y) and the Cauchy-Riemann equations are satisfied then $f(z) = u(x, y) + iv(x, y)$ is complex differentiable at $z = x + iy$.

3. If $f'(z) = 0$ in a region D then f is constant on D.

4. If $|f(z)| < M$ on a curve C then $\left\| \int_C f(z)dz \right\| < ML$ where L ia the length of the curve.

5. The following statements are equivalent:
   (i) f has an antiderivative F;
   (ii) $\int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1)$;
   (iii) If C is a closed curve then $\int_C f(z)dz = 0$.

6. Cauchy-Goursat Theorem (outline). If f is analytic on and inside a simple closed curve C then $\int_C f(z)dz = 0$.

7. If f is analytic in the region between closed curves C_2 and C_1 with C_1 inside C_2 then
$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz.$$

8. The Cauchy Integral Formula.

9. A bounded entire function is constant.

10. If f is analytic on annulus, it equals its Laurent series (outline).

11. Cauchy Residue Theorem.

Sample Exercises:

1. Let
$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & \text{for } x \neq 0; \\ \alpha, & \text{for } x = 0, \end{cases}$$
where $\alpha \in [-1, 1]$.

   (a) Is f continuous?

   (b) Does f have the intermediate value property?

2. Let
$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & \text{for } x \neq 0; \\ 0, & \text{for } x = 0, \end{cases}$$

   (a) Show that f is differentiable everywhere.

   (b) Is f' continuous?

3. Given that f is a quadratic polynomial
$$f(x) = Kx^2 + Lx + M,\ K \neq 0,$$
show that the point c whose existence is guaranteed by the Mean Value Theorem is the midpoint of the interval.

4. Let
$$f(x) = \begin{cases} 0, & \text{for } x \leq 0; \\ \sin\left(\frac{\pi}{x}\right), & \text{for } x > 0, \end{cases} \quad \text{and } g(x) = \begin{cases} 0, & \text{for } x \leq 0; \\ 1, & \text{for } x > 0, \end{cases}$$
Prove that f has an antiderivative but g does not.

5. On continuity.

   (a) State the definition of continuity for maps $f : X \to Y$, in which X and Y are topological spaces, X with prescribed topology $\mathcal{T}_X$, and Y with prescribed topology $\mathcal{T}_Y$. Explain how this intrinsic definition of continuity generalizes the $\epsilon - \delta$ definition learned in Calculus 1

   (b) Design topologies on $\mathbb{R}$ with respect to which the floor function $x \mapsto [x]$ is continuous

   (c) In the *natural topology* on $\mathbb{R}$, denoted $\mathcal{T}_{nat}$, a set $\mathcal{O} \subset \mathbb{R}$ is said to be open if
$$\forall x \in \mathcal{O} \quad \exists \delta > 0 \quad \text{such that } (x - \delta, x + \delta) \subset \mathcal{O}\,.$$
Show that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous according to the $\epsilon - \delta$ definition of continuity learned in Calculus 1 *if and only if* $f^{-1}(V)$ (the inverse image under f of $V \subset \mathbb{R}$) is open in $\mathcal{T}_{nat}$ whenever V is open in $\mathcal{T}_{nat}$.

6. On uniform convergence.

   (a) Show that if a sequence $\{f_n\}$ of continuous functions converges uniformly on a domain $\Omega \subset \mathbb{R}$ to a function f, then the limit f is also continuous on Ω.

   (b) Let $\{f_n\}$ be a sequence of continuously differentiable functions such that $\{f_n\}$ and $\{f_n'\}$ converge uniformly on a domain Ω to the limiting functions f and g, respectively. Show that for every x in the interior of Ω,

$$g(x) \equiv \lim_{n\to\infty} f_n'(x) = \left(\lim_{n\to\infty} f_n(x)\right)' \equiv f'(x).$$

7. State Ascoli-Arzela's Theorem and outline its proof.

8. The sequence of continuous functions $\{f_n : [0, 2\pi] \to \mathbb{R}\}_{n\in\mathbb{N}}$ with f_n given by $f_n(x) = \sin(nx)$ is uniformly bounded, but not equicontinuous. Give an intuitive reason why such a sequence is not equicontinuous, then give a rigorous proof.

9. Compute the following integral limit

$$\lim_{n\to\infty} \int_{\mathbb{R}} \frac{n \sin(x/n)}{x(x^2+1)} dx$$

10. Consider the function

$$f(z) = \begin{cases} \frac{\bar{z}^2}{z}, & \text{if } z \neq 0; \\ 0, & \text{if } z = 0;. \end{cases}$$

    Is this function differentiable at $z = 0$? Is it continuous at $z = 0$?

11. Consider an entire function $f : \mathbb{C} \to \mathbb{C}$, where $f(z) = u(x, y) + iv(x, y)$, and $z = x + iy$. Show that, if $u(x, y) \leq x$ for all x, y, then f is of the form $f(z) = z + c$. (Hint: Use Liouville theorem. Consider the function $g(z) = e^{-z+f(z)}$.)

12. Compute

$$\int_C \frac{1}{1+z^2} dz$$

    where C is the circle of radius 3 centered at the origin.

13. Compute

$$\int_C \frac{1}{\sin(z)} dz$$

    over the contour C shown in the Figure 1.

14. Suppose f is an analytic function on the punctured unit disk $D \setminus \{0\}$ such that $f(1/n) = (-1)^n/n$ for all positive integers n. Prove that $\lim_{z\to 0} |f(z)|$ does not exist.

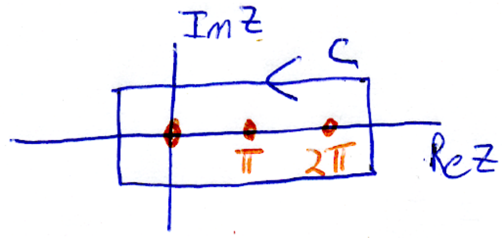

Figure 1: Contour C

15. Show that the only conformal maps from the complex plane onto itself are the non-constant linear maps, i.e. maps of the form $f(z) = az + b$, $a \neq 0$.

16. Let f be a *doubly periodic function*, that is, there are two complex numbers w_1, w_2 with $w_1/w_2 \notin \mathbb{R}$ so that for any $z \in \mathbb{C}$, $f(z) = f(z+w_1) = f(z+w_2)$. Let us also assume that f is meromorphic.

    (a) Show that if f is an entire function, then it has to be constant.

    (b) Let Γ be the boundary of the parallelogram with vertices $0, w_1, w_2, w_1+w_2$, oriented counterclockwise. Show that if f is analytic on Γ, then $\int_\Gamma f(z)dz = 0$.

    (c) Assuming that f is analytic on Γ and has exactly one singularity inside Γ, show that the residue at this singularity is necessarily zero.

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