Quasiperiodic Noise For Bifurcation Mapping And Optimization: Numerical Study of Voltage Output in a Nonlinear Energy Harvester

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Poster synthesized from the uploaded semester summary, figures, and template. Core method references were added in APA style to complete the bibliography.

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Introduction

Background. A nonlinear piezoelectric harvester is a forced magnetoelastic Duffing oscillator with electrical coupling [Moon & Holmes, 1979; Erturk & Inman, 2011]. Its double-well potential creates coexisting attractors and strong sensitivity to initial conditions (ICs).

Problem with a single seed. A bifurcation diagram swept in ω from one IC samples only the basin reached by that seed; entire branches—including ones that carry the highest mean-square voltage $\langle w^2 \rangle$—stay hidden, biasing both physical interpretation and parameter tuning.

State of the art. Recent work shows that the highest energy output in such systems comes from coexistence of chaotic and regular attractors rather than from pure resonance [Gidea et al.; Akingbade et al., 2023]. Quasiperiodic (QP) forcing is the natural setting for that picture and is supported by Arnold-diffusion theory.

Research challenges. (i) Recover a more complete bifurcation portrait without enumerating every basin; (ii) regularize the multi-modal $\langle w^2 \rangle(\omega)$ objective so derivative-free / Bayesian search converges; (iii) keep the perturbation physically realistic—multi-frequency, low-amplitude.

Aim. Inject a small QP noise n(t) and test whether it (a) makes the bifurcation diagram more complete, (b) yields a smoother $\langle w^2 \rangle(\omega)$ for ω-optimization, and (c) preserves the high-voltage solutions.

Approach

Model (single-beam piezoelectric harvester). $\dot{x} = y$, $\dot{y} = ½x(1-x^2) - 2\zeta y + \chi w + \varepsilon \sin((\omega + n(t))\cdot t)$, $\dot{w} = -\lambda w - \kappa y$. x: dimensionless beam displacement; w: dimensionless harvested voltage; ω: excitation frequency (design variable). Fixed: $\zeta = 0.01$ (damping), $\chi = 0.05$, $\kappa = 0.5$ (electromechanical coupling), $\varepsilon = 0.115$ (forcing amplitude), $\lambda \in \{0.01, 0.05\}$ (electrical time constant).

Quasiperiodic perturbation. $n(t) = A[\cos(f_1 t) + \cos(f_2 t) + \cos(f_3 t)]$, $A = 10^{-4}$, $f_1 = 1$, $f_2 = (\sqrt{5} - 1)/2 \approx 0.618$, $f_3 = \sqrt{769} - 27 \approx 0.730$ (incommensurate → no resonant lock-in).

Numerical pipeline. RK45 (rtol = 10^{-8}, atol = 10^{-9}); 1000 transient periods discarded, 2000 stroboscopic samples ($T = 2\pi/\omega$) retained per ω. ICs probed: (1, 0, 0), (0.56, 0.429, 0.16), (−0.2, 0.62, 0.47); broken branches recovered by forward/backward continuation.

Optimization problem. Maximize the smoothed mean-square voltage $J(\omega) = \langle w^2 \rangle = (1/N) \Sigma\, w(nT)^2$ over $\omega \in [0.5, 2.0]$, with $T = 2\pi/\omega$. Black-box, derivative-free, expensive single-objective problem (one stiff ODE per evaluation; QP forcing makes J noisy). Solved with (a) Nelder–Mead simplex (local) and (b) TuRBO trust-region Bayesian optimization (global, GP surrogate + Thompson sampling) [Eriksson et al., 2019].

Quasiperiodic perturbation: $n(t) = A[\cos(f_1 t) + \cos(f_2 t) + \cos(f_3 t)]$ with $A = 10^{-4}$, $f_1 = 1$, $f_2 = (\sqrt{5}-1)/2$, $f_3 = \sqrt{769}-27$

Scan initial conditions › **Build deterministic diagram** › **Add QP perturbation and compare** › **Smooth objective & optimize ω**

Outcomes

Metric: $\langle w^2 \rangle$ averaged over 2000 stroboscopic samples; symmetric IC $x_0 = (0.56, 0.429, 0.16)$ recovers the broken branch and serves as the "complete" deterministic baseline (Fig. 1).
QP-noise effect: a 10^{-4} perturbation crosses basin boundaries, exposes missing branches, and broadens the high-$\langle w^2 \rangle$ plateau in $0.7 \le \omega \le 1.1$ (Figs. 2–3).
Smoothing: Gaussian filtering regularizes $\langle w^2 \rangle(\omega)$ without erasing the plateau (Fig. 4).
Benchmark & insight: TuRBO converges to $\omega^* \approx 0.94$, $\lambda^* \approx 0.02$, $\langle w^2 \rangle \approx 0.21$ in ~400 evaluations (Fig. 5). QP forcing acts as a low-energy basin-hopping regularizer — no input power penalty.

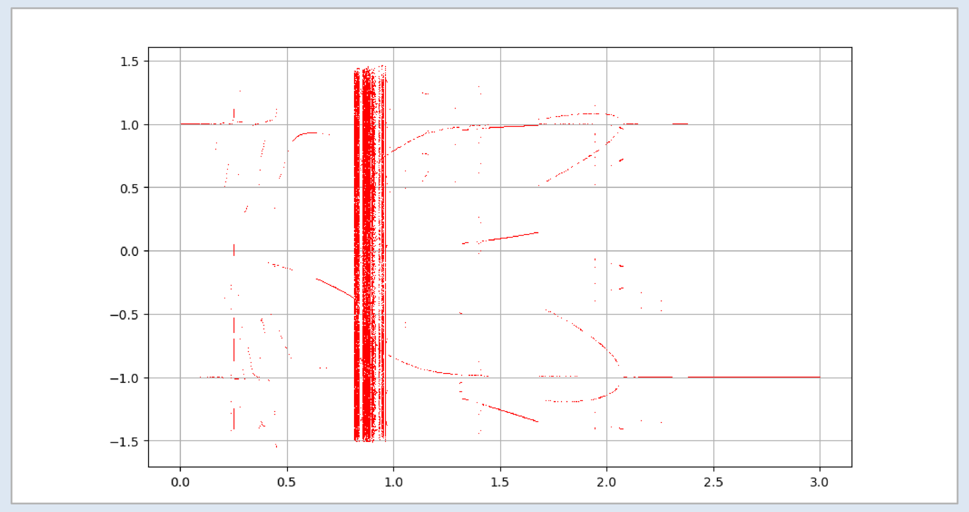

Fig. 1. Deterministic bifurcation, $x_0 = (0.56, 0.429, 0.16)$.

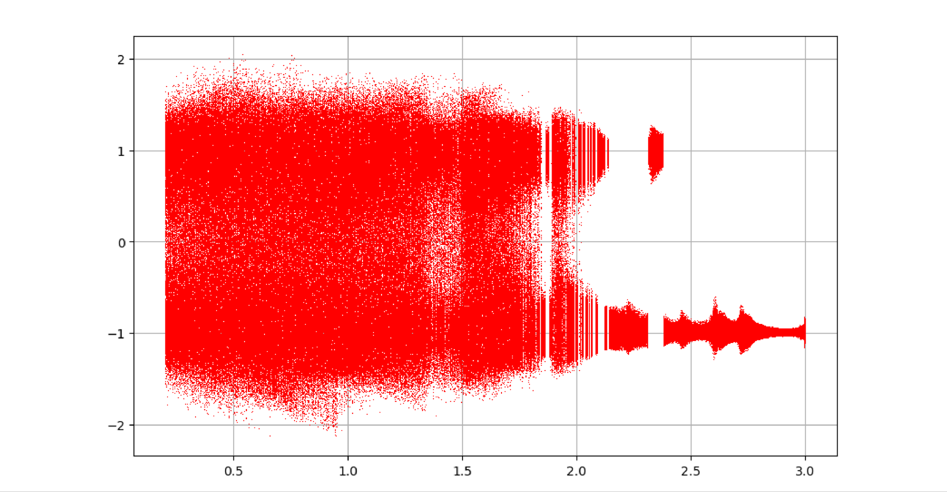

Fig. 2. Quasiperiodic-noise bifurcation, same x_0.

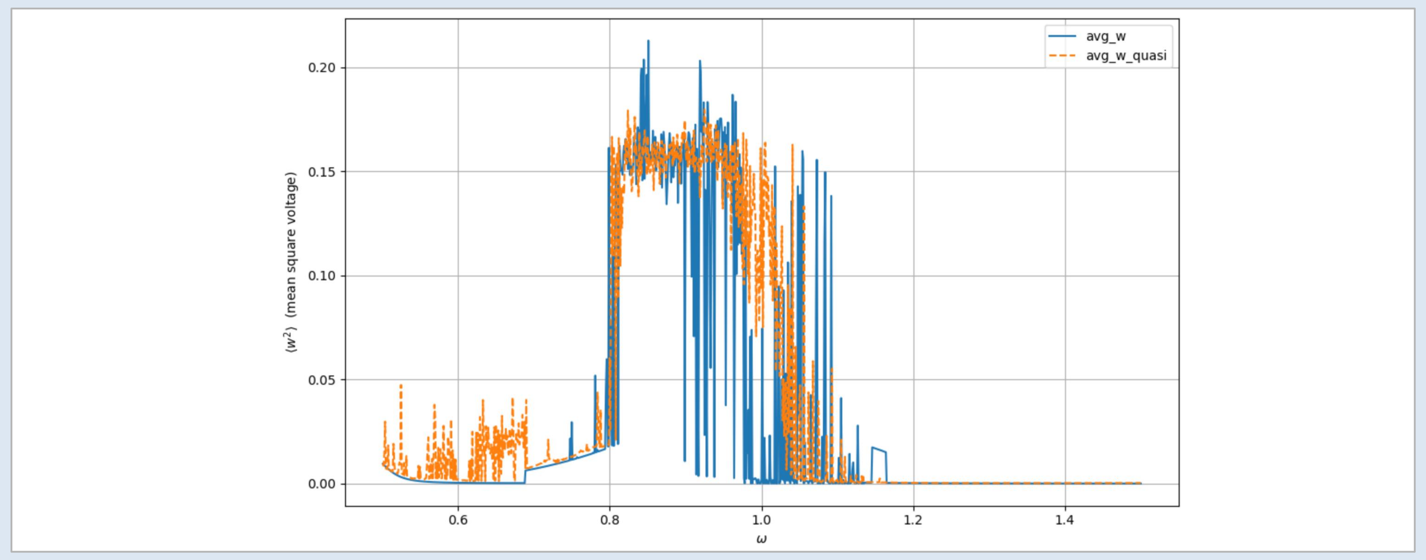

Fig. 3. $\langle w^2 \rangle$ comparison with vs without quasiperiodic noise.

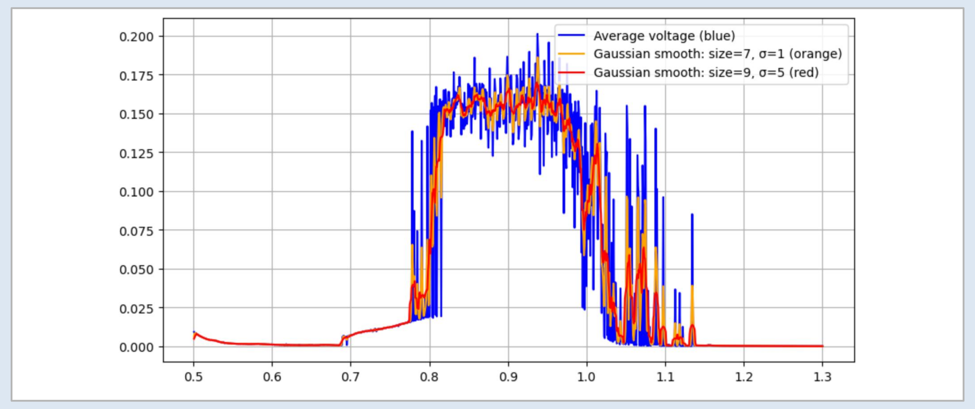

Fig. 4. Gaussian smoothing of average voltage $\langle w^2 \rangle(\omega)$.

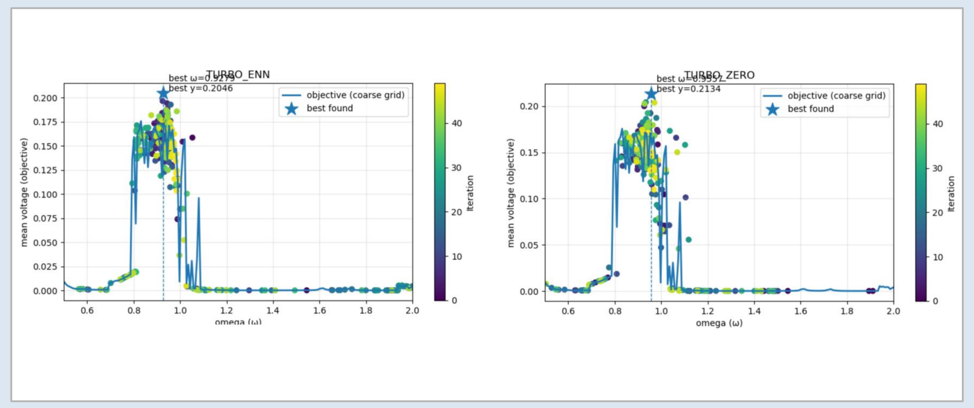

Fig. 5. TuRBO optimization trajectories (zero / ENN) under noise.

Conclusions

- **C1. Multiple ICs are essential.** A single seed misses high-$\langle w^2 \rangle$ branches; a symmetric IC recovers them (Fig. 1).
- **C2. QP noise improves bifurcation completeness.** Missing branches re-emerge and the high-$\langle w^2 \rangle$ region broadens (Figs. 2, 3).
- **C3. QP noise + Gaussian smoothing yields a search-friendly objective.** Local oscillations in $\langle w^2 \rangle(\omega)$ are damped (Fig. 4), enabling stable TuRBO / Nelder–Mead convergence (Fig. 5).
- **C4. Performance is preserved.** Best solutions under noise ($\omega^* \approx 0.94$, $\lambda^* \approx 0.02$, $\langle w^2 \rangle \approx 0.21$) match the deterministic optimum—QP forcing acts as a regularizer with no power penalty.

Limitations & Next Steps:

- Results are based on a small set of initial conditions and one noise amplitude.
- The source summary did not report formal statistical testing or a full project bibliography.
- Next steps: scan amplitudes/frequencies systematically, quantify bifurcation completeness, and benchmark optimization across denser ω grids.

References

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