Saturday, November 3, 2018 11:36 PM

⇒ Let us assume that our matrices are real symmetric. In this case and also when the matrix is Hermitian, the eigenvalues are REAL.

$$\begin{cases} \text{Real symmetric} \Rightarrow & A_{ij} = A_{ji} \\ \text{Hermitian} \Rightarrow & A_{ij}^* = A_{ji} \end{cases}$$

⇒ For a matrix A and a vector V satisfying

$$\boxed{A.V = \lambda V}$$

we say that V is the eigenvector of A and λ is its corresponding eigenvalue.

For an $N \times N$ matrix, there are N eigenvectors of dimension N and N corresponding eigenvalues

⇒ Assuming that $V \neq 0$, we can write the equation above as

$$(A - \mathbb{1}\lambda) V = 0$$

identity matrix

If the whole matrix in front of V has an inverse

$$(A - \mathbb{1}\lambda)^{-1} (A - \mathbb{1}\lambda) V = V = 0,$$

which goes against our assumption.

So $(A - \mathbb{1}\lambda)$ does <u>NOT</u> have an inverse, which also means that the <u>determinant</u>

$$|A - \mathbb{1}\lambda| = 0$$

This is what we can use to find the eigenvalues

⇒ Example

$$A = \begin{pmatrix} 1 & 2 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = 0$$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4+12}}{2} \quad \boxed{\lambda_1 = 3} \quad \boxed{\lambda_2 = -1}$$

⇒ Finding the eigenvectors

$$A \cdot V_1 = \lambda_1 V_1$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 3 \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{cases} x + 2y = 3x \\ 2x + y = 3y \end{cases}$$

From the equations above

$$y = x$$

so $V_1 = \begin{pmatrix} x \\ x \end{pmatrix}$

We can now get x from the conditions of NORMALIZATION

that is $|V_1| = 1$

$$\hookrightarrow |x|^2 + |y|^2 = 1$$

$$\Downarrow$$

$$2|x|^2 = 1 \Rightarrow x = \frac{1}{\sqrt{2}}$$

so $\boxed{V_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$ is the eigenvector associated with eigenvalue $\boxed{\lambda_1 = 3}$

How about $V_2 = ?$

$$A \cdot V_2 = \lambda_2 V_2$$

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -1 \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{cases} x + 2y = -x \\ 2x + y = -y \end{cases}$$

From above $y = -x$

$$V_2 = \begin{pmatrix} x \\ -x \end{pmatrix}$$

From $|V_2| = 1$

$$|x|^2 + |-x|^2 = 1$$

$$x = \frac{1}{\sqrt{2}}$$

$$V_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$ is the eigenvector associated with eigenvalue $\lambda_2 = -1$