LU decomposition

If we want to solve many different sets of equations

\[ A \mathbf{x} = \mathbf{v} \]

with the same matrix \( A \) but different \( \mathbf{v} \)'s, it would be more efficient to do the full Gaussian elimination just once and record the divisions, multiplications, and subtractions, so that later we can repeat it to any \( \mathbf{v} \).

This is what we call LU decomposition.

Using Python numbering, suppose we have the matrix

\[
A = \begin{pmatrix}
a_{00} & a_{01} & a_{02} & a_{03} \\
a_{10} & a_{11} & a_{12} & a_{13} \\
a_{20} & a_{21} & a_{22} & a_{23}
\end{pmatrix}
\]
The first step of the Gaussian elimination is to divide 1st row by $a_{00}$, so that the new 1st element $b_{00} = 1$.

\[ \text{2) (2nd row)} - a_{20} \text{ (new 1st row)}, \]
so that $b_{20}$ becomes 0.

\[ \text{3) (3rd row)} - a_{20} \text{ (new 1st row)}, \]
so that $b_{20} = 0$.

\[ \text{4) (4th row)} - a_{30} \text{ (new 1st row)}, \]
so that $b_{30} = 0$.

This can be accomplished with a single matrix multiplication,

\[ L_0 \cdot A = \begin{pmatrix} 1 & b_{01} & b_{02} & b_{03} \\ 0 & 1 & b_{12} & b_{13} \\ 0 & 0 & 1 & b_{23} \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

where

\[ \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix} \]
where
\[
L_0 = \frac{1}{a_{00}} \begin{pmatrix}
1 & 0 & 0 & 0 \\
\frac{-a_{10}}{a_{00}} & a_{00} & 0 & 0 \\
\frac{-a_{20}}{a_{00}} & 0 & a_{00} & 0 \\
\frac{-a_{30}}{a_{00}} & 0 & 0 & a_{00}
\end{pmatrix}
\]

(1) CHECK that this is true!

Note: \( L_0 \) is a lower triangular matrix

(2) The second step of the Gaussian elimination

1. divide 2nd row by \( b_{21} \) so \( c_{11} = 1 \)

2. (3rd row) - \( b_{21} \) (new 2nd row) so \( c_{21} = 0 \)

3. (4th row) - \( b_{31} \) (new 2nd row) so \( c_{31} = 0 \)

\[
L_1 \cdot B = \begin{pmatrix}
1 & b_{01} & b_{02} & b_{03} \\
0 & 1 & c_{12} & c_{13} \\
0 & 0 & c_{22} & c_{23} \\
0 & 0 & c_{32} & c_{33}
\end{pmatrix}
\]
where

\[ L_1 = \frac{1}{b_{11}} \begin{pmatrix} b_{11} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -b_{21} & b_{11} & 0 \\ 0 & -b_{31} & 0 & b_{11} \end{pmatrix} \]

(3) Third step

\[ L_2 = \frac{1}{c_{22}} \begin{pmatrix} c_{22} & 0 & 0 & 0 \\ 0 & c_{22} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -c_{32} & c_{22} \end{pmatrix} \]

(4) Fourth step

\[ L_3 = \frac{1}{d_{33}} \begin{pmatrix} d_{33} & 0 & 0 & 0 \\ 0 & d_{33} & 0 & 0 \\ 0 & 0 & d_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \]

Therefore, the complete Gaussian elimination
on $A$ is equivalent to

$$L_3 L_2 L_1 L_0 A = L_3 L_2 L_1 L_0 U$$

this is an upper-triangular new vector matrix (which is the goal of the Gauss elimination)

The equation above can be solved with backsubstitution exactly as we did before.

The advantage now is that we have

$$L_3 L_2 L_1 L_0 A \text{ stored}$$
and

$$L_3 L_2 L_1 L_0 \text{ stored}$$

and can solve the system of equations for any $v$ avoiding having to compute $L_3 L_2 L_1 L_0 A$ again.

In brachy, however, the calculation is a
In practice, however, the calculation is a little different.

We define

\[
L = L_0^{-1} \cdot L_1^{-1} \cdot L_2^{-1} \cdot L_3^{-1} \quad \text{and} \quad U = L_3 \cdot L_2 \cdot L_1 \cdot L_0 \cdot A
\]

for lower triangular matrix \( \Rightarrow \) for upper triangular matrix

\[
L \cdot U = A
\]

\( \Leftrightarrow \) LU decomposition of matrix A

We can indeed verify that

\[
L_0^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
\frac{a_{10}}{a_{00}} & 1 & 0 & 0 \\
\frac{a_{20}}{a_{10}} & 0 & 1 & 0 \\
\frac{a_{30}}{a_{20}} & 0 & 0 & 1
\end{pmatrix} \quad L_1^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & b_{11} & 0 & 0 \\
0 & b_{21} & 1 & 0 \\
0 & b_{31} & 0 & 1
\end{pmatrix}
\]

\[
L_2^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & c_{22} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \quad L_3^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]
\[
\begin{pmatrix}
0 & 0 & c_{22} & 0 \\
0 & 0 & c_{32} & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & d_{33}
\end{pmatrix}
\]

\[
L_3^{-1} L_2^{-1} L_1^{-1} L_0^{-1} = 
\begin{pmatrix}
a_{00} & 0 & 0 & 0 \\
a_{10} & b_{11} & 0 & 0 \\
a_{20} & b_{21} & c_{22} & 0 \\
a_{30} & b_{31} & c_{32} & d_{33}
\end{pmatrix}
\]

Our set of equations can be written as

\[
A \chi = \nu
\]

\[
L U \chi = \nu
\]

Taking as example a 3x3 case

\[
\begin{pmatrix}
l_{00} & 0 & 0 \\
l_{10} & l_{11} & 0 \\
l_{20} & l_{21} & l_{22}
\end{pmatrix}
\]

\[
\begin{pmatrix}
m_{00} & m_{01} & m_{02} \\
m_{10} & m_{11} & m_{12} \\
m_{20} & 0 & m_{22}
\end{pmatrix}
\begin{pmatrix}
\chi_0 \\
\chi_1 \\
\chi_2
\end{pmatrix}
= 
\begin{pmatrix}
\nu_0 \\
\nu_1 \\
\nu_2
\end{pmatrix}
\]
we define the vector $y = \begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}$ as $Ux = y$.

which can be obtained by

**BACK SUBSTITUTION**

in $L \cdot y = v$

After obtaining $y$, we turn to $U \cdot x = y$ to obtain $x$ also by

**BACK SUBSTITUTION**