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# LU decomposition

If we want to solve many different sets of equations

$$A x = v$$

with the same matrix A but different $v$'s, it would be more efficient to do the full Gaussian elimination just once and record the divisions, multiplications, and subtractions, so that later we can repeat it to any $v$.

This is what we call LU decomposition.

Using Python numbering, suppose we have the matrix

$$A = \begin{pmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{10} & a_{11} & a_{12} & a_{13} \\ a_{20} & a_{21} & a_{22} & a_{23} \end{pmatrix}$$

$$A = \begin{pmatrix} a_{20} & a_{21} & a_{22} & a_{23} \\ a_{30} & a_{31} & a_{32} & a_{33} \end{pmatrix}$$

① The first step of the Gaussian elimination is

- to divide 1st row by $a_{00}$, so that the new 1st element $b_{00} = 1$
- (2nd row) $- a_{10}$ (new 1st row), so that $b_{10}$ becomes 0.
- (3rd row) $- a_{20}$ (new 1st row), so that $b_{20} = 0$
- (4th row) $- a_{30}$ (new 1st row), so that $b_{30} = 0$

This can be accomplished with a single matrix multiplication

$$L_0 \cdot A = \underbrace{\begin{pmatrix} 1 & b_{01} & b_{02} & b_{03} \\ 0 & b_{11} & b_{12} & b_{13} \\ 0 & b_{21} & b_{22} & b_{23} \\ 0 & b_{31} & b_{32} & b_{33} \end{pmatrix}}_{B}$$

where

$$\begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$$

where

$$L_0 = \frac{1}{a_{00}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -a_{10} & a_{00} & 0 & 0 \\ -a_{20} & 0 & a_{00} & 0 \\ -a_{30} & 0 & 0 & a_{00} \end{pmatrix}$$

<u>CHECK</u> that this is true!

<u>Note</u>: $L_0$ is a <u>lower triangular</u> matrix

(2) The second step of the Gaussian elimination

- .) divide 2nd row by $b_{11}$ so $c_{11} = 1$
- .) (3rd row) − $b_{21}$ (new 2nd row) so $c_{21} = 0$
- .) (4th row) − $b_{31}$ (new 2nd row) so $c_{31} = 0$

$$L_1 \cdot B = \underbrace{\begin{pmatrix} 1 & b_{01} & b_{02} & b_{03} \\ 0 & 1 & c_{12} & c_{13} \\ 0 & 0 & c_{22} & c_{23} \\ 0 & 0 & c_{32} & c_{33} \end{pmatrix}}_{C}$$

where

$$L_1 = \frac{1}{b_{11}} \begin{pmatrix} b_{11} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -b_{21} & b_{11} & 0 \\ 0 & -b_{31} & 0 & b_{11} \end{pmatrix}$$

③ Third step

$$L_2 = \frac{1}{c_{22}} \begin{pmatrix} c_{22} & 0 & 0 & 0 \\ 0 & c_{22} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -c_{32} & c_{22} \end{pmatrix}$$

④ Fourth step

$$L_3 = \frac{1}{d_{33}} \begin{pmatrix} d_{33} & 0 & 0 & 0 \\ 0 & d_{33} & 0 & 0 \\ 0 & 0 & d_{33} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Therefore, the complete Gaussian elimination on A is equivalent to

$$\underbrace{L_3 . L_2 . L_1 . L_0 . A}_{\substack{\text{this is an} \\ \text{UPPER-TRIANGULAR}}} = \underbrace{L_3 . L_2 . L_1 . L_0 . v}_{\substack{\text{this is a} \\ \text{new vector}}}$$

this is an UPPER-TRIANGULAR matrix (which is the goal of the Gauss elimination) — this is a new vector

The equation above can be solved with

BACKSUBSTITUTION

exactly as we did before.

The advantage now is that we have

$\boxed{L_3 \, L_2 \, L_1 \, L_0 \, A}$ stored

and

$\boxed{L_3 \, L_2 \, L_1 \, L_0}$ stored

and can solve the system of equations for any $v$ avoiding having to compute $L_3 . L_2 . L_1 . L_0 . A$ again

!!

In practice, however, the calculation is a

In practice, however, the calculation is a little different.

We define

$$L = L_0^{-1} . L_1^{-1} . L_2^{-1} . L_3^{-1} \quad \text{and} \quad U = L_3 L_2 L_1 L_0 A$$

for lower triangular matrix

for upper triangular matrix

$$\boxed{L . U = A}$$

LU decomposition of matrix A

We can indeed verify that

$$L_0^{-1} = \begin{pmatrix} a_{00} & 0 & 0 & 0 \\ a_{10} & 1 & 0 & 0 \\ a_{20} & 0 & 1 & 0 \\ a_{30} & 0 & 0 & 1 \end{pmatrix} \qquad L_1^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & b_{11} & 0 & 0 \\ 0 & b_{21} & 1 & 0 \\ 0 & b_{31} & 0 & 1 \end{pmatrix}$$

$$L_2^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c_{22} & 0 \end{pmatrix} \qquad L_3^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & c_{22} & 0 \\ 0 & 0 & c_{32} & 1 \end{pmatrix} \qquad \begin{pmatrix} \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & d_{33} \end{pmatrix}$$

$$L = L_0^{-1} L_1^{-1} L_2^{-1} L_3^{-1} = \begin{pmatrix} a_{00} & 0 & 0 & 0 \\ a_{10} & b_{11} & 0 & 0 \\ a_{20} & b_{21} & c_{22} & 0 \\ a_{30} & b_{31} & c_{32} & d_{33} \end{pmatrix}$$

Our set of equations can be written as

$$\underbrace{A}\, x = v$$

$$L\, U\, x = v$$

Taking as example a $3\times 3$ case

$$\underbrace{\begin{pmatrix} l_{00} & 0 & 0 \\ l_{10} & l_{11} & 0 \\ l_{20} & l_{21} & l_{22} \end{pmatrix}}_{L} \cdot \underbrace{\underbrace{\begin{pmatrix} u_{00} & u_{01} & u_{02} \\ 0 & u_{11} & u_{12} \\ 0 & 0 & u_{22} \end{pmatrix}}_{U} \cdot \underbrace{\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}}_{x}} = \underbrace{\begin{pmatrix} v_0 \\ v_1 \\ v_2 \end{pmatrix}}_{v}$$

we define the vector $y$ $\begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix}$

as $\boxed{U.x = y}$

which can be obtained by

**BACK SUBSTITUTION**

in $\boxed{L.y = v}$

After obtaining $y$, we turn to

$\boxed{U.x = y}$ to obtain $x$

also by

**BACK SUBSTITUTION**