Static and dynamic properties of a one-dimensional spin-1/2 system

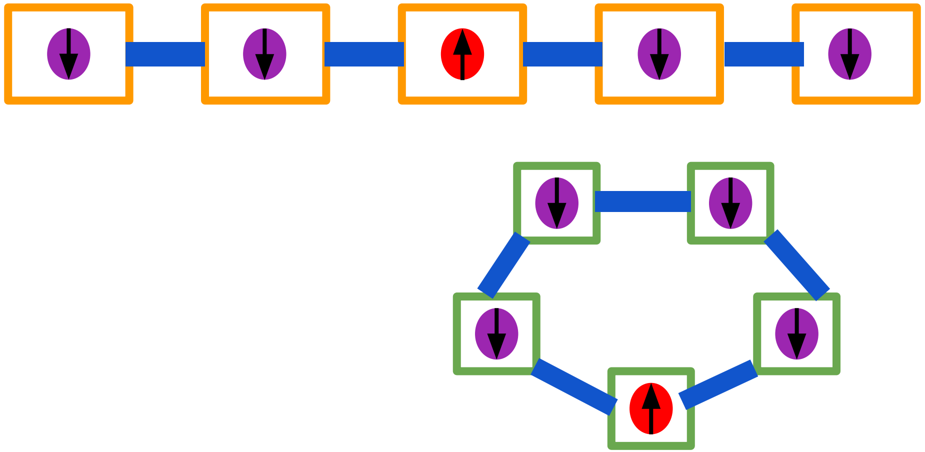
Aviva Shooman and Lea Santos
Yeshiva University, Stern College for Women

Overview

- Description of a 1D spin ½ model
- Study of its time evolution (dynamics)
- Illustration of the spread of an excitation through the system

1D spin ½ Lattice System

- On each site there is a spin 1/2 (up or down in z direction)
- Chain or ring formations determine geometry between sites



Chain

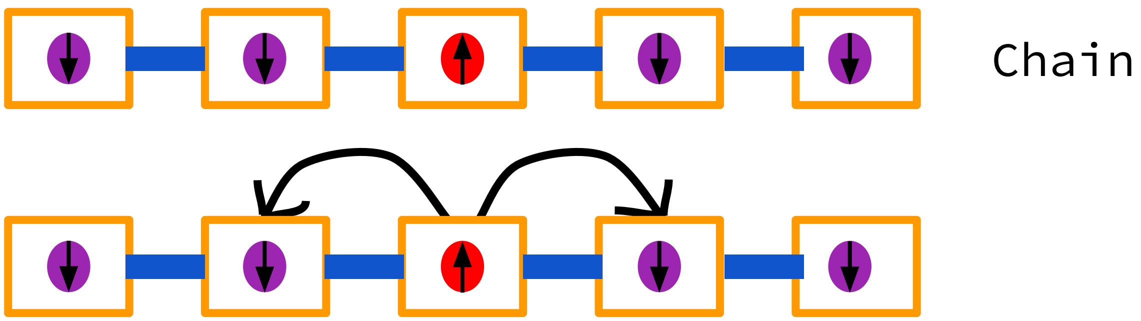
Ring

$$\uparrow = \begin{bmatrix}1\\0\end{bmatrix}$$

$$\downarrow = \begin{bmatrix}0\\1\end{bmatrix}$$

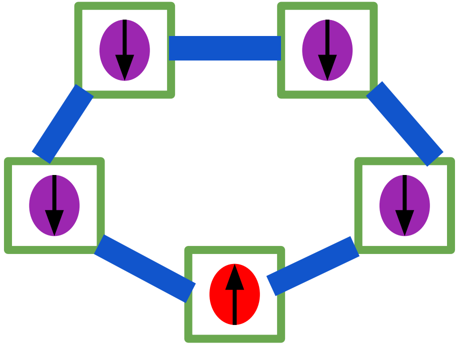
Chain Formation

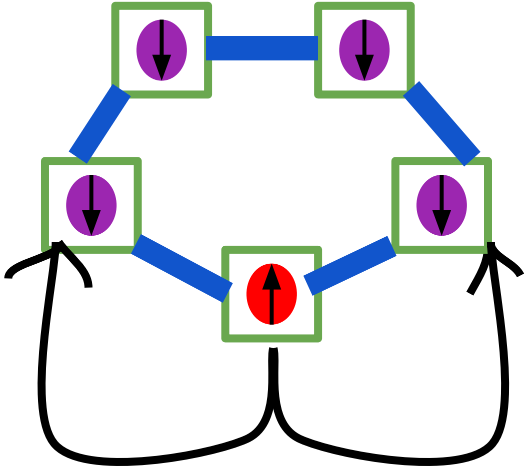
- Particle pointing upwards is the excitation
- Interactions happen between neighboring sites
- As a result the excitation hops to either side of the neighboring site



Ring Formation

- Interactions happen between neighboring sites and the two endpoints(last and first)
- The excitation hops to either side of the neighboring site





Ring

Spin ½ Hamiltonian

- J is the coupling strength and gives the energy scale
- L is the number of sites
- The first term gives the energy of each site
- The second is called the Flip-Flop term because it moves the excitation along the chain

$$\hat{H} = \sum_{n=1}^{L} \left[\frac{\varepsilon_n}{2} \sigma_n^z\right] + \frac{J}{2} \sum_{n=1}^{L-1} \left[\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y\right]$$

Pauli Matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

- The Pauli matrices in the x and y direction flip the spin and the z gives the energy of each site (down becomes negative)

$$\frac{\varepsilon_1}{2}\sigma_1^z|\uparrow\rangle = \frac{\varepsilon_1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = +\frac{\varepsilon_1}{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = +\frac{\varepsilon_1}{2}|\uparrow\rangle$$

$$\frac{\varepsilon_1}{2}\sigma_1^z|\downarrow\rangle = \frac{\varepsilon_1}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\frac{\varepsilon_1}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\frac{\varepsilon_1}{2}|\downarrow\rangle$$

Sigma x, Pauli Matrix x

- The sigma x flips the spin

$$\sigma^x|\uparrow> \Rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |\downarrow>$$

$$\sigma^x|\downarrow> \Rightarrow \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |\uparrow>$$

Sigma y, Pauli Matrix y

- The sigma y flips the spin and has an additional i term

$$\sigma^y | \uparrow > \Rightarrow \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = +i \begin{bmatrix} 0 \\ 1 \end{bmatrix} = +i \,| \downarrow >$$

$$\sigma^y | \downarrow > \Rightarrow \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -i \begin{bmatrix} 1 \\ 0 \end{bmatrix} = -i \,| \uparrow >$$

Flip-Flop Term

- The Flip-Flop term moves the excitation along the chain

$$\frac{J}{2}\left(\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y\right) | \uparrow\downarrow > = \frac{J}{2} | \downarrow\uparrow >$$

$$\frac{J}{2}\left(\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y\right) | \downarrow\uparrow > = \frac{J}{2} | \uparrow\downarrow >$$

Site Basis-Vectors

$$\uparrow\downarrow\downarrow\downarrow,\ \downarrow\uparrow\downarrow\downarrow,\ \downarrow\downarrow\uparrow\downarrow,\ \downarrow\downarrow\downarrow\uparrow$$

- L=4 with 1 excitation
- States where on each site the spin either points up or down (in the z direction)
- The Zeeman energies of all sites are equal, $\varepsilon_n = \varepsilon$

$$\downarrow \quad \uparrow \quad \downarrow \quad \downarrow$$

$$-\cancel{\frac{\varepsilon_1}{2}} + \cancel{\frac{\varepsilon_2}{2}} \underbrace{-\frac{\varepsilon_3}{2} - \frac{\varepsilon_4}{2}}_{-\varepsilon}$$

Hamiltonian in a Matrix Form

- Diagonal elements are the sum the energies of each site
- Off-diagonal elements are produced by the Flip-Flop term
- Tridiagonal Matrix (real and symmetric)

Example: open chain with
L=4 sites and 1 excitation

	↑↓↓↓	↓↑↓↓	↓↓↑↓	↓↓↓↑
↑↓↓↓	$-\varepsilon$	$J/2$	0	0
↓↑↓↓	$J/2$	$-\varepsilon$	$J/2$	0
↓↓↑↓	0	$J/2$	$-\varepsilon$	$J/2$
↓↓↓↑	0	0	$J/2$	$-\varepsilon$

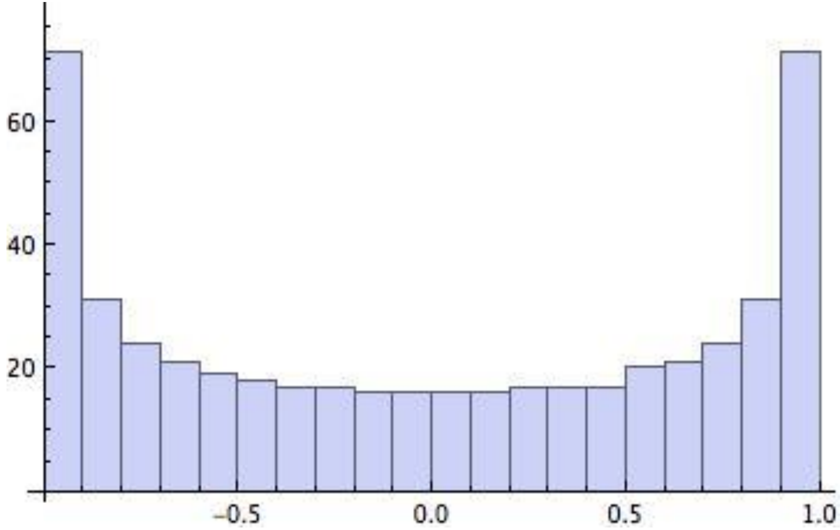
Eigenvalues and Eigenstates

- We diagonalize the Hamiltonian matrix to get all eigenvalues, E_n and eigenstates, $|\psi_n>$
- For each energy level (eigenvalue) there is one corresponding eigenstate
- Each eigenstate is a superposition of several site-basis vectors

Density of States

- L=20 and 1 excitation

Numerical:



Analytical:

$$\rho(\varepsilon) = \frac{1}{\pi\sqrt{J^2 - \varepsilon^2}}$$

Spreading of the Excitation through the Sites

- Initially, we place the excitation on the first site. Later in time it has a probability of being found on other sites of the chain
- The spreading of an excitation along the chain is a typical property of quantum mechanics

Schrödinger's equation:

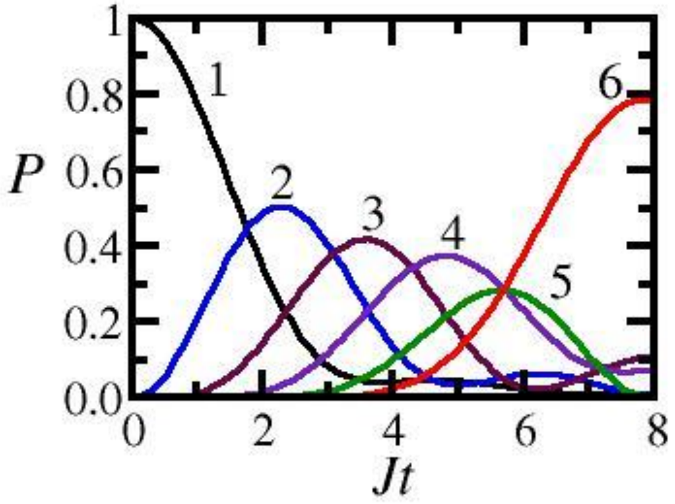
$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H}\Psi$$

Evolution of the state:

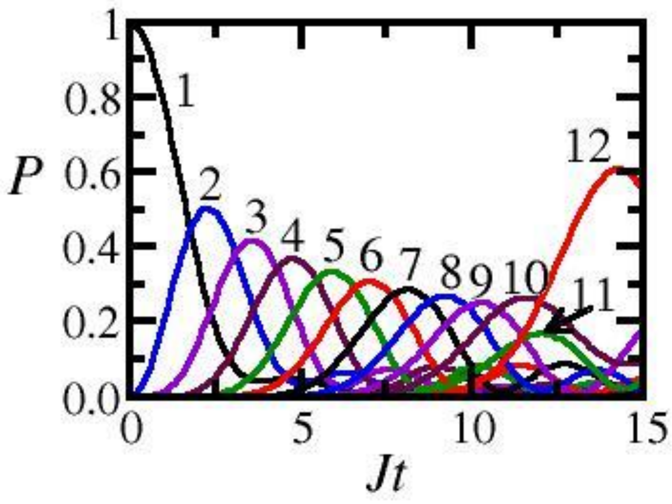
$$|\Psi(t)> = \sum_{n=1}^{L} C_n e^{-iE_n t} |\psi_n>$$

Spreading of the Excitation through the Sites

L=6



L=12



Conclusion

- Spin ½ systems are prototype quantum many-body systems.
- They can be used to introduce undergraduate students to the properties of quantum mechanics and a variety of current subjects of interest, such as: quantum phase transition, metal-insulator transition, quantum chaos, thermalization, and quantum computing.