The Absolute Roll Measure

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Abstract

Roll (1984) and Thompson and Waller (1987) provide techniques for estimation of bid-ask spreads from mid-market closing prices. Restricting to this limited information, I introduce the Absolute Roll Measure derived using imaginary numbers to address the confounding presence of positive autocovariance in Roll (1984). I conduct an empirical implementation for Aaa and Baa corporate bond spreads over the period 1986-2020 with four bid-ask models. The Absolute Roll Measure resolves an issue in the literature and provides a measure applicable to all traded securities that are limited to closing price information.

Key Words: Corporate Bonds; Liquidity; Microstructure
JEL Codes: C0, G10, G12, Z0

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Introduction

Central to empirical investigations into the information content of securities transactions, their price formation, and their liquidity is the bid-ask spread. The bid-ask spread represents the difference between the quotes of where market-makers stand ready to buy (bid-side) and sell (ask-side aka 'offered-side') securities at a specific point in time and provides a rich measure of liquidity as discussed in Amihud (2002). The absence of observable bid-ask spreads for asset classes (due to infrequent recording) may compound their risk opacity. This is often the case with infrequently traded securities as noted in Hasbrouck (2009) and Fong, Holden and Trzcinka (2017). Sparser information, in turn, can exacerbate illiquidity particularly in times of distress as discussed in Bao, O'Hara and Zhou (2018).

As an absolute value moving average estimator, the Thompson and Waller (1987) model is one model that is well-defined for this purpose as discussed in He and Mizrach (2017). It provides bid-ask spreads for all such end of day mid-market observations. In contrast, the more sophisticated model of Roll (1984), which uses autocovariance of price changes, has the well-known limitation of validity only for observations that exhibit negative autocovariance between asset returns. This is because instances of positive autocovariance result in complex numbers in the model formulation as noted in Lo and Wang (2000), and are thus undefined. As I discuss, existing adaptations in the literature to Roll (1984) that are restricted to closing prices (and no other data) result in either negative bid-ask spreads or omission of large amounts of observations as observed in several studies. This is dissatisfying and motivates this work.

In this paper I address this issue by providing a new adaptation of Roll (1984) which is derived using imaginary numbers, and is defined as the *Absolute Roll Measure*, $\hat{s} = 2\sqrt{|-\mathrm{COV}\left(\Delta P_t, \Delta P_{t+1}\right)|}$. This new measure of the effective bid-ask spread of securities is fully generalizable and applicable to *any* traded asset, yielding strictly non-negative bid-ask spreads. I empirically implement the Absolute Roll Measure, as well as three existing approaches in the microstructure literature, to estimate daily bid-ask spreads for triple-A (Aaa) rated and triple-B (Baa) US corporate bonds over the sample period 1/13/1986 - 7/1/2020. The initial results of the new methodology expand on an existing approach and resolves in existing issue in the literature. This should allow for more robust testing of liquidity across asset classes, particularly those which do not have the benefit of intraday pricing. Additionally, the contribution of this paper would allow for unfettered estimation of bid/ask spreads intraday to be compared to more sophisticated models as discusseed in O'Hara (1997) that develop following Roll (1984). This is left to future research.

The remainder of this paper is organized as follows: Section 1 discusses the data used in this study. Section 2 positions the paper within the literature. Section 3 derives the Absolute Roll Measure, while Section 4 provides the empirical implementation of bid-ask estimation using the various models described. Section 5 provides statistical summary results for the models. Section 6 concludes with suggestions for future work. The Online Appendix provides supplementary information.

1 Data

In this section I discuss the data used throughout this study. I use 8629 daily time series from the Federal Reserve Bank of St. Louis FRED system for the empirical implementations. The sample period is 1/13/1986 thru 7/1/2020. These data include corporate bond credit risk premia for Aaa and Baa corporate bond ratings. The credit risk premia on Aaa and Baa corporate bonds are in excess of the 10 year constant maturity rate (CMT) US Treasury yields. CMT yields thus represent the implied cost of new borrowing for a given maturity at a given point in time. By extension, Aaa and Baa risk premia in excess of 10-year CMT yields are also constant maturity and serve as a benchmark cost of new borrowing for a given maturity and credit at a point in time.

2 Review of issues in the literature

This section considers some of the literature pertaining to bid-ask spreads determined from closing prices.

2.1 Roll (1984)

The seminal work on bid-ask spreads derived from closing end-of-day prices, P_t , is found in Roll (1984).[1] Briefly, as discussed in Lo and Wang (2000), solving for the effective bid-ask spread, s

$$\begin{aligned} -\frac{s^2}{4} =& \mathrm{COV}\left(\Delta P_t, \Delta P_{t+1}\right) \\ s^2 =& -4\mathrm{COV}(\Delta P_t, \Delta P_{t+1}) \\ \sqrt{s^2} =& \sqrt{-4\mathrm{COV}(\Delta P_t, \Delta P_{t+1})} \end{aligned}$$

$$s = 2\sqrt{-\mathrm{COV}(\Delta P_t, \Delta P_{t+1})} \tag{1}$$

yields a complex number when the first order autocovariance $\mathrm{COV}\left(\Delta P_t, \Delta P_{t+1}\right) > 0$.

Roll (1984) and Harris (1990) address this problem by treating the value of s differently in different domains. Following Harris (1990), values for s^+ apply to instances of positive autocovariance, while values for s^- apply to instances of negative autocovariance, such that:

$$s = \begin{cases} s^+ = -2\sqrt{\mathrm{COV}(\Delta P_t, \Delta P_{t+1})} & \text{for } \mathrm{COV}(\Delta P_t, \Delta P_{t+1}) > 0 \\ s^- = 2\sqrt{-\mathrm{COV}(\Delta P_t, \Delta P_{t+1})} & \text{for } \mathrm{COV}(\Delta P_t, \Delta P_{t+1}) \leq 0 \end{cases} \tag{2}$$

This choice has the effect, as noted in Lo and Wang (2000) of 'preserving the sign of the covariance' in keeping with the empirical analyses of Roll (1984) and Harris (1990). However this preservation of sign also results in negative effective bid-ask spread estimates, s, as shown in Eq. (2). This is dissatisfying. Negative bid-ask spreads imply market-makers inverting markets; standing ready to buy securities at *higher* prices than where they would sell them. Such providing of liquidity would be ruinous and thus unrealistic.

One way to address this issue within the Roll (1984) framework is simply to drop (or 'zero') observations of positive autocovariance as suggested by Hasbrouck (2009) and Foucault, Pagano and Roell (2013). This too is unsatisfying given the large number of observations that exhibit instances of positive autocovariance. Since instances of positive autocovariance are frequently observed in ranges of 28% to as much as 50% in prior studies, this

[1] Derived, following Harris (2003), in Section A.1 of the Online Appendix.

issue warrants further investigation in this paper.

2.2 Thompson and Waller (1987)

Finally, an alternative bid-ask spread estimator model restricted to end of day closing prices is introduced in Thompson and Waller (1987). Like Roll (1984), Thompson and Waller (1987) also use end-of-day prices to estimate bid-ask spreads from the absolute value of 5-day moving averages of changes in end-of-day closing prices, ($\overline{|\Delta p_t|} = \frac{1}{5}\sum_{t=1}^{5} \Delta p_t$). The model is practical in that it guarantees a strictly non-negative bid-ask estimator result. It is also actively utilized in policy and by practitioners, as discussed in He and Mizrach (2017).

3 Absolute Roll Measure

In this section I introduce a novel adaptation of Roll (1984) using imaginary numbers.

Claim. I claim the use of imaginary numbers within the radicand of Eq. (1) allows a new measure for the effective bid-ask spread estimate based on Roll (1984) referred to as the *Absolute Roll Measure*, and defined as:

$$\hat{s} = 2\sqrt{|-\mathrm{COV}\left(\Delta P_t, \Delta P_{t+1}\right)|} \tag{3}$$

Definition. Following Roll (1984), let the autocovariance of asset returns, s, be defined[2] such that

$$s = \begin{cases} s^{+} = 2\sqrt{-\mathrm{COV}(\Delta P_t, \Delta P_{t+1})} & \text{for } \mathrm{COV}(\Delta P_t, \Delta P_{t+1}) > 0 \\ s^{-} = 2\sqrt{-\mathrm{COV}(\Delta P_t, \Delta P_{t+1})} & \text{for } \mathrm{COV}(\Delta P_t, \Delta P_{t+1}) < 0 \\ s^{z} = 2\sqrt{-\mathrm{COV}(\Delta P_t, \Delta P_{t+1})} & \text{for } \mathrm{COV}(\Delta P_t, \Delta P_{t+1}) = 0 \end{cases} \tag{4}$$

with the imaginary number, i, defined[3] as

$$i = \sqrt{-1} \Longleftrightarrow i^2 = -1 \tag{5}$$

[2]This is motivated by Harris (1990) to 'preserve the sign' of the autocovariance, but carries different signs for s^{+} inside and outside the radical.

[3]See Simon and Blume (1994).

Conjecture. For s^+,

$$\begin{aligned} s^+ =& 2\sqrt{\text{-COV}(\Delta P_t, \Delta P_{t+1})} \\ =& 2\sqrt{\text{-1}\times\underbrace{\text{COV}\left(\Delta P_t, \Delta P_{t+1}\right)}_{>0}} \\ =& 2\underbrace{\sqrt{-1}}_{=i}\sqrt{\underbrace{\text{COV}\left(\Delta P_t, \Delta P_{t+1}\right)}_{>0}} \\ =& i \times \underbrace{2\sqrt{\text{COV}(\Delta P_t, \Delta P_{t+1})}}_{=y} \end{aligned}$$

Let $x = 0$ and $y = 2\sqrt{\underbrace{\text{COV}(\Delta P_t, \Delta P_{t+1})}_{>0}}$, so by complex numbers

$$s^+ = yi$$

with the complex conjugate of s^+ defined as

$$\overline{s^+} = -yi$$

and their product

$$\begin{aligned} s^+\overline{s^+} =& -y^2\underbrace{i^2}_{=-1} \\ =& -y^2 \times -1 \\ =& y^2 \end{aligned}$$

Taking the square root of both sides

$$\begin{aligned} \underbrace{\sqrt{s^+\overline{s^+}}}_{=|s^+|} =& \sqrt{y^2} \\ \left|s^+\right| =& y \\ =& 2\sqrt{\underbrace{\text{COV}(\Delta P_t, \Delta P_{t+1})}_{>0}} \end{aligned}$$

$$\therefore \left|s^+\right| = 2\sqrt{\underbrace{\text{COV}(\Delta P_t, \Delta P_{t+1})}_{>0}} \tag{6}$$

which is the distance of s^+ from the origin, in the complex plane.

Similarly, for s^-,

$$
\begin{aligned}
s^{-} =& 2\sqrt{\text{-COV}(\Delta P_t, \Delta P_{t+1})} \\
=& 2\sqrt{\text{-}1\times\underbrace{\text{COV}(\Delta P_t, \Delta P_{t+1})}_{<0}} \\
=& 2\sqrt{\text{-}1\times\underbrace{-1\times\text{COV}\left(\Delta P_t, \Delta P_{t+1}\right)}_{<0}} \\
=& 2\sqrt{\underbrace{-1\times-1}_{=1}\times\underbrace{\text{COV}\left(\Delta P_t, \Delta P_{t+1}\right)}_{>0}} \\
=& 2\sqrt{1\times\underbrace{\text{COV}\left(\Delta P_t, \Delta P_{t+1}\right)}_{>0}}
\end{aligned}
$$

$$
=2\sqrt{\underbrace{\text{COV}(\Delta P_t, \Delta P_{t+1})}_{>0}} = \left|s^{-}\right| \tag{7}
$$

by absolute values, which is the distance of the real number, s^{-} from the origin.

Finally, for s^{z},

$$
\begin{aligned}
s^{z} =& 2\sqrt{\text{-COV}(\Delta P_t, \Delta P_{t+1})} \\
=& 2\sqrt{\text{-}1\times\underbrace{\text{COV}(\Delta P_t, \Delta P_{t+1})}_{=0}} \\
=& 2\sqrt{\text{-}1\times\underbrace{-1\times\text{COV}\left(\Delta P_t, \Delta P_{t+1}\right)}_{=0}} \\
=& 2\sqrt{\underbrace{-1\times-1}_{=1}\times\underbrace{\text{COV}\left(\Delta P_t, \Delta P_{t+1}\right)}_{=0}} \\
=& 2\sqrt{1\times\underbrace{\text{COV}\left(\Delta P_t, \Delta P_{t+1}\right)}_{=0}}
\end{aligned}
$$

$$
=2\sqrt{\underbrace{\text{COV}(\Delta P_t, \Delta P_{t+1})}_{=0}} = |s^{z}| \tag{8}
$$

by absolute values, which is the distance of the real number, s^{z} from the origin, at the origin, which is zero.

Solution. Eqs. (6 and 7) yield the result that $|s^{+}| = |s^{-}|$. The radicands for both Eqs. (6 and 7) are shown to be $\text{COV}\left(\Delta P_t, \Delta P_{t+1}\right) > 0$. Post the evaluation with imaginary numbers above and on the domain partitions in Eq. (4), for each of those radicands

$\mathrm{COV}(\Delta P_t, \Delta P_{t+1}) = |\mathrm{COV}(\Delta P_t, \Delta P_{t+1})|$ allowing for substitution of $|\mathrm{COV}(\Delta P_t, \Delta P_{t+1})|$ for $\mathrm{COV}(\Delta P_t, \Delta P_{t+1})$ in the radicands for Eqs. (6, 7 and 8) allowing them to be restated as:

$$\left|s^+\right| = 2\sqrt{|\mathrm{COV}(\Delta P_t, \Delta P_{t+1})|} \text{for } \mathrm{COV}(\Delta P_t, \Delta P_{t+1}) > 0 \tag{9}$$

$$\left|s^-\right| = 2\sqrt{|\mathrm{COV}(\Delta P_t, \Delta P_{t+1})|} \text{for } \mathrm{COV}(\Delta P_t, \Delta P_{t+1}) < 0 \tag{10}$$

$$|s^z| = 2\sqrt{|\mathrm{COV}(\Delta P_t, \Delta P_{t+1})|} \text{for } \mathrm{COV}(\Delta P_t, \Delta P_{t+1}) = 0 \tag{11}$$

Since Eqs. (9, 10, and 11) yield the same value, it follows that

$$\left|s^+\right| = \left|s^-\right| = |s^z| = 2\sqrt{|\mathrm{COV}(\Delta P_t, \Delta P_{t+1})|} \text{ for } -\infty < \mathrm{COV}(\Delta P_t, \Delta P_{t+1}) < \infty \tag{12}$$

But

$$2\sqrt{|\mathrm{COV}(\Delta P_t, \Delta P_{t+1})|} = 2\sqrt{|-\mathrm{COV}(\Delta P_t, \Delta P_{t+1})|} \text{ for } -\infty < \mathrm{COV}(\Delta P_t, \Delta P_{t+1}) < \infty \tag{13}$$

Therefore,

$$|s| = 2\sqrt{\underbrace{|-\mathrm{COV}(\Delta P_t, \Delta P_{t+1})|}_{\geq 0}} = s \tag{14}$$

The absolute value of a non-negative number is simply the number itself. Since the radicand in Eq. (14) is strictly non-negative, Eq. (14) equals Eq. (3), and thus $|s| = s = \hat{s}$ for *all* $\mathrm{COV}(\Delta P_t, \Delta P_{t+1})$ through the use of imaginary numbers. As such, $\hat{s}$ is the *Absolute Roll Measure*, an adaptation of the effective spread of Eq. (1) with imaginary numbers. □

This Absolute Roll Measure, $\hat{s}$, guarantees a strictly non-negative bid-ask spread observable for all price changes in all traded asset markets. The Absolute Roll Measure, $\hat{s}$, is the magnitude of s, regardless of whether s is real or imaginary. For observations where $\mathrm{COV}(\Delta P_t, \Delta P_{t+1}) > 0$, the interpretation is the corresponding effective bid-ask spread exists in the complex plane.

4 Model implementation

In this section I summarize an implementation of the four models above. I depict their implementation for Aaa corporate bond risk premia in Figure 1 over the sample period. The x-axis in the plots captures the range of values for the autocovariance while the y-axis captures the range of values for the effective bid-ask spreads.

These time series of corporate bond risk premia are widely followed fixed-income benchmarks. Since corporate bonds trade on a risk premium (spread) basis to corresponding maturity risk-free rates I substitute the daily mark-to-market risk premia (aka 'Spreads'), S_t, in place of prices, P_t, for this estimate of the bid-ask spreads which, like P_t , are i.i.d. This allows for the calculation of bid/ask spreads of Spreads, corresponding to how the instruments are traded.

It is true that even with constant maturity (no aging, no cashflows) the modified duration of the bonds will be different in each period by definition.[4] As such, changes in risk premia (as proxies for prices) would be technically valid only for one period in this time series. However, in this initial study differences across the sample period vary only by about 1.50% over thirty-four years. As such, in this initial study, for brevity I assume constant modified durations across all time periods, allowing for changes in Spreads to be the object of inquiry. This is reasonable. for constant maturity treasuries (and by implication, constant maturity corporates) are constant across observations, autocovariances of changes to either risk premia, S_t, or prices, P_t, adjusted for bond modified durations will yield the identical values for bid-ask spreads under Roll (1984). This observation supports the assumption permitting a direct assessment of fixed-income credit products which almost always trade on a Spread to risk-free basis (and not a price basis). Future research can make further adjustments for the durations corresponding to the instruments.

Since Roll (1984) and its adaptations (Models 1, 2 and 3) necessarily have the same autocovariances, differences in depictions are only associated with differences in the model adjustment choices. Although Model 4 (the moving average method of Thompson and Waller (1987)) does not use autocovariance, since all autocovariances are temporal, I show Model 4

[4] SeeFabozzi (2007).

bid-ask spreads corresponding to the same observation dates as the autocovariances for the other three models. The autocovariances are used in Models 1, 2 and 3 are calculated over a 5-day (1 trading week) period. Model 4 is also captured over a 5-day (1 trading week) period.

4.1 Model summaries

In Model 1 (Roll), I implement Roll (1984) for end-of-day mid-market spread risk premia. such that

$$s_t = 2\sqrt{-\mathrm{COV}(\Delta S_t, \Delta S_{t-1})} \tag{15}$$

To address positive autocovariance in the implementation I follow Harris (1990) in Eq. (2)

$$s_t = \begin{cases} s_t^+ = -2\sqrt{\mathrm{COV}(\Delta S_t, \Delta S_{t-1})} & \text{for } \mathrm{COV}(\Delta S_t, \Delta S_{t-1}) > 0 \\ s_t^- = 2\sqrt{-\mathrm{COV}(\Delta S_t, \Delta S_{t-1})} & \text{for } \mathrm{COV}(\Delta S_t, \Delta S_{t-1}) \leq 0 \end{cases} \tag{16}$$

which preserves the sign of autocovariance. Figure 1a captures Model 1 (Roll). There we indeed observe negative bid-ask spreads corresponding to positive autocovariances.

In Model 2 (Restricted Roll) I restrict Model 1, by simply dropping (or 'zeroing') the observations with positive autocovariance as suggested by Hasbrouck (2009) and Foucault, Pagano and Roell (2013). Figure 1b shows an implementation of Model 2.[5]

In Model 3 (Absolute Roll), the Absolute Roll Measure for risk premia is given by

$$\hat{s}_t = 2\sqrt{\left|-\mathrm{COV}\left(\Delta S_t, \Delta S_{t-1}\right)\right|} \tag{17}$$

following the derivation in Section 3, culminating in Eq. (14). Figure 1c shows the implementation of Model 3 in Eq. (17). We see a strictly non-negative bid-ask spread for all Aaa observations, as expected. The positive autocovariance values to the right of the origin on the x-axis in Figure 1c exhibit positive bid-ask spreads on the y-axis in contrast to Figure

[5] In the statistical summary (available upon request) I provide both the zeroed version and the dropped versions for convenience. Note that in any statistical analysis, only the restricted version dropping values with positive autcovariance should be used, as artificial 'zeroing' could produce misleading statistical results.

1a and Figure 1b.

In Model 4 (Thompson and Waller), I implement the model of Thompson and Waller (1987) and restate their absolute value of 5-day moving average price changes using mid-market credit spreads as

$$\overline{|\Delta S_t|} = \frac{1}{5}\sum_{t=1}^{5} \Delta S_t \tag{18}$$

The bid-side spread is given by

$$B_t = S_t + \left(\frac{\overline{|\Delta S_t|}}{2}\right) \tag{19}$$

and the ask-side spread[6] is given by

$$A_t = \max\left(0,\ S_t - (B_t - S_t)\right) \tag{20}$$

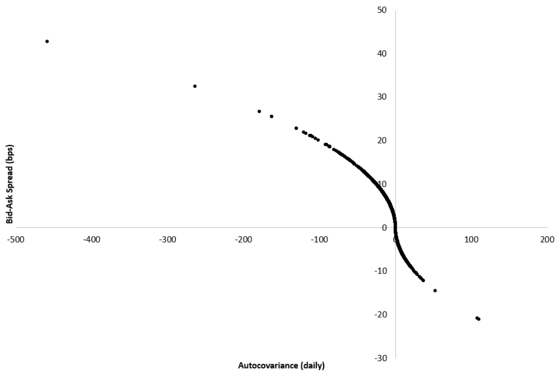
such that the bid-ask spread is given by

$$s_t = B_t - A_t \tag{21}$$

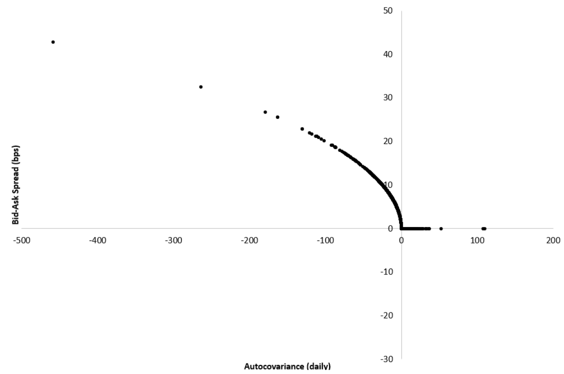
Figure 1d shows the implementation of Model 4. It is interesting to see the contrast between the Absolute Roll Measure depicted in Figure 1c and the model of Thompson and Waller (1987) depicted in Figure 1d.

[6] The lower boundary of zero ensures offer side spreads cannot be negative. The boundary is never reached in this study.

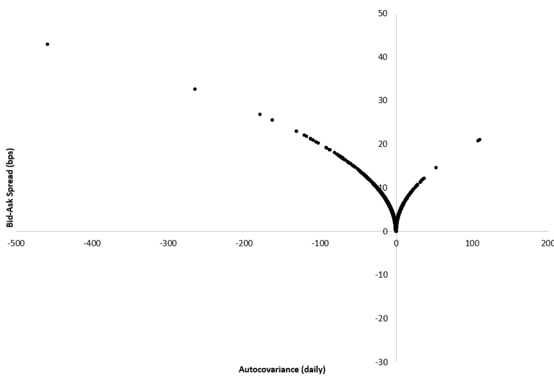
Figure 1: Aaa bid-ask estimates



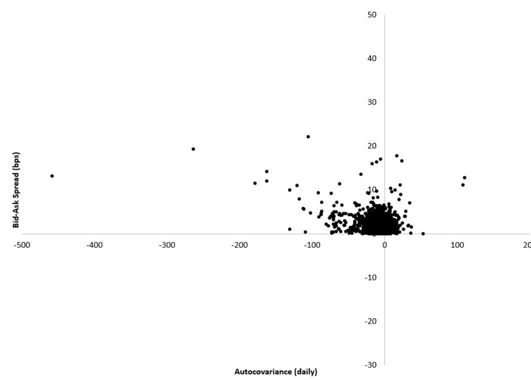
(a) Model 1: Roll (1984)



(b) Model 2: Restricted Roll (1984)



(c) Model 3: Absolute Roll



(d) Model 4: Thompson-Waller

While both models guarantee strictly non-negative bid-ask spreads, the underlying processes yield different results.[7] I discuss this further below.

[7] Section A.2 in the Online Appendix shows the time series comparisons for Models 3 and 4 for Aaa and Baa credits.

5 Statistical summary

Table 1: Statistical summary: Aaa

	Mid Market	Model 1 Roll	Model 2 RstrctRoll (zeroed)	Model 2 RstrctRoll	Model 3 AbsRoll	Model 4 TW
mean	135.54	1.70	2.35	3.25	3.00	1.07
median	134.00	1.74	1.74	2.53	2.33	0.80
min	31.00	-20.98	0.00	0.00	0.00	0.00
max	320.00	42.83	42.83	42.83	42.83	22.20
var	2154.70	12.84	7.73	7.76	6.69	1.48
stdev	46.42	3.58	2.78	2.79	2.59	1.22
obs	8629	8623	8623	6240	8623	8623

Table 2: Statistical summary: Baa

	Mid Market	Model 1 Roll	Model 2 RstrctRoll (zeroed)	Model 2 RstrctRoll	Model 3 AbsRoll	Model 4 TW
mean	233.46	1.51	2.18	3.04	2.85	1.15
median	221.00	1.55	1.55	2.30	2.15	0.80
min	116.00	-32.01	0.00	0.00	0.00	0.00
max	616.00	26.14	26.14	26.14	32.01	20.80
var	5151.09	12.38	7.15	7.37	6.56	1.84
stdev	71.77	3.52	2.67	2.71	2.56	1.36
obs	8629	8623	8623	6188	8623	8623

Consistent with the earlier findings of Roll (1984), Harris (1990) and Hasbrouck (2009) I also find positive autocovariance to be frequently observable in this study. In my sample of 8623 autocovariance observations of Aaa and Baa corporate bond risk premia, 2383 (27.64%) of the Aaa observations and 2435 (28.24%) of the Baa observations exhibited positive autocovariance. Use of Model 1 would result in a substantial number of negative bid-ask spreads, while Model 2 would result in substantial elimination of observations for those percentages over the sample period. In contrast, Models 3 and 4 provide strictly non-negative effective bid-ask spread estimates. Table 1 and Table 2 provide statistical summaries for the observed mid-market risk premia and the Model determined bid-ask spreads, for Aaa and Baa corporate credit risk premia, respectively. A few interesting observations from those tables are worth highlighting.

Model 1, for example shows negative minimum bid-ask spreads for Aaa and Baa, while

Model 2 shows minimum bid-ask spreads of zero. Additionally, median statistics of Models 1 and 2 are identical. These findings are as expected. Conventional wisdom for the mid-market observations is confirmed with Baa credits exhibiting greater and more volatile risk premia than Aaa credits.

Interestingly, this is not echoed in the effective bid-ask spreads produced by the models. Models 1, 2 and 3 show Baa bid-ask spreads to be less volatile than Aaa credits. This contrasts with the statistics for Model 4, which may be due to differences in methodologies. Further, Model 4 exhibits categorically lower bid-ask maximums. Comparing Models 3 and 4, I find Model 3 exhibits wider bid-ask spreads than Model 4 in 88.01% and 84.52% of the cases for Aaa and Baa securities, respectively.[8]

Perhaps most interesting is the comparison among the maximums for bid-ask spreads across credit ratings. Categorically, across all models, Aaa credits exhibit wider maximum bid-ask spreads than Baa credits. The comparatively more 'fragile' and less liquid characteristic found in Aaa credits is a surprising and rich finding. These findings should motivate future research.

6 Summary

The introduction of the Absolute Roll Measure in this paper addresses a confounding issue in the microstructure literature. The new method addresses the issue of positive autocovariance in Roll (1984) and eliminates the need to drop large numbers observations as it produces strictly non-negative bid-ask spreads as is also found in Thompson and Waller (1987) which yields different results. This paper provides a robust and practical method for liquidity assessment in the microstructure context where pricing information is limited. Expansions of this approach with more recent developments in the microstructure literature that rely upon intraday data may also be made with this methodology given adequate pricing information, intraday. Overall, the Absolute Roll Measure may increase our understanding of liquidity, particularly in those sectors that are limited to end of day pricing information such as many instruments in the US structured finance market such as CMBS, CMBX, CRE-CDOs and

[8] Section A.2 in the Online Appendix shows the time series comparisons for Models 3 and 4 for Aaa and Baa credits.

others which is left to future work.

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A Online Appendix for 'The Absolute Roll Measure'

A.1 Derivation of Roll (1984) following Harris (2003)

For any time t, let P_t be the observed price and V_t the fundamental price (aka 'fair-value') that follows a random walk with ϵ_t an i.i.d. white noise process $\sim N(0, \sigma^2)$. The temporal indicator of 'Buy' or 'Sell' of an asset from the market maker's perspective, Q_t,is thus defined as

$$Q_t = \begin{cases} +1, & \text{'Buy'with Probability} = p = \left(\frac{1}{2}\right) \\ -1, & \text{'Sell'with Probability} = 1 - p = \left(\frac{1}{2}\right) \end{cases} \tag{22}$$

Taking expectations,

$$E[Q_t] = \left(\frac{1}{2}\right)(1) + \left(\frac{1}{2}\right)(-1) = 0 \tag{23}$$

and

$$E[Q_t^2] = \left(\frac{1}{2}\right)(1^2) + \left(\frac{1}{2}\right)(-1^2) = 1 \tag{24}$$

The market price, is composed of the fair value and the 'effective' bid-ask spread, S, such that,

$$P_t = V_t + Q_t \left(\frac{1}{2}\right) S \tag{25}$$

with changes in market prices given by

$$\Delta P_t = \underbrace{\Delta V_t}_{=\epsilon_t} + \Delta Q_t \left(\frac{1}{2}\right) S \tag{26}$$

which implies

$$\Delta P_t = \Delta Q_t \left(\frac{1}{2}\right) S + \epsilon_t \tag{27}$$

The *mean* of the change in price is determined by

$$
\begin{aligned}
E[\Delta P_t] =& E\left[\Delta Q_t\left(\frac{1}{2}\right)S+\epsilon_t\right] \\
=& E\left[\Delta Q_t\left(\frac{1}{2}\right)S\right]+E\left[\underbrace{\epsilon_t}_{=0}\right] \\
=& \left(\frac{1}{2}\right)SE\left[\Delta Q_t\right] \\
=& \left(\frac{1}{2}\right)SE\left[Q_t-Q_{t-1}\right] \\
=& \left(\frac{1}{2}\right)S\left\{\underbrace{E\left[Q_t\right]}_{=0}-\underbrace{E\left[Q_{t-1}\right]}_{=0}\right\} \\
=& 0
\end{aligned}
\tag{28}
$$

The *variance* of the change in price is determined by

$$
\begin{aligned}
Var[\Delta P_t] =& E\left[\Delta Q_t\left(\frac{1}{2}\right)S+\epsilon_t\right] \\
=& E\left[\left(\Delta Q_t\left(\frac{1}{2}\right)S+\epsilon_t\right)^2\right] \\
=& E\left[\epsilon_t^2+2\epsilon_t\Delta Q_t\left(\frac{1}{2}\right)S+\Delta Q_t^2\left(\frac{1}{4}\right)S^2\right] \\
=& E\left[\epsilon_t^2\right]+E\left[2\epsilon_t\Delta Q_t\left(\frac{1}{2}\right)S\right]+E\left[\Delta Q_t^2\left(\frac{1}{4}\right)S^2\right] \\
=& \underbrace{E\left[\epsilon_t^2\right]}_{=\sigma_\epsilon^2}+S\underbrace{E\left[\epsilon_t\Delta Q_t\right]}_{=0}+\left(\frac{1}{4}\right)S^2E\left[\Delta Q_t^2\right] \\
=& \sigma_\epsilon^2+\left(\frac{1}{4}\right)S^2E\left[\left(Q_t-Q_{t-1}\right)^2\right] \\
=& \sigma_\epsilon^2+\left(\frac{1}{4}\right)S^2E\left[Q_t^2-2Q_tQ_{t-1}+Q_{t-1}^2\right] \\
=& \sigma_\epsilon^2+\left(\frac{1}{4}\right)S^2\left\{\underbrace{E\left[Q_t^2\right]}_{=1}-E\left[2Q_tQ_{t-1}\right]+\underbrace{E\left[Q_{t-1}^2\right]}_{=1}\right\} \\
=& \sigma_\epsilon^2+\left(\frac{1}{4}\right)S^2\left\{1-2\underbrace{E\left[Q_t\right]}_{=0}\underbrace{E\left[Q_{t-1}\right]}_{=0}+1\right\} \\
=& \sigma_\epsilon^2+\left(\frac{1}{2}\right)S^2
\end{aligned}
\tag{29}
$$

All Q_{t-k} are independent $\therefore E\left[Q_tQ_{t-k}\right]=E\left[Q_t\right]E\left[Q_{t-k}\right]=0$ for $k>0$. As such, the *autocovariance* of serial changes in price, $\text{COV}\left(\Delta P_t,\Delta P_{t-1}\right)$, is determined by

$$
\begin{aligned}
\mathrm{COV}\left(\Delta P_t, \Delta P_{t-1}\right) =& E\left[\left(\Delta P_t - \underbrace{E\left[\Delta P_t\right]}_{=0}\right)\left(\Delta P_{t-1} - \underbrace{E\left[\Delta P_{t-1}\right]}_{=0}\right)\right] \\
=& E\left[\left(\Delta P_t\right)\left(\Delta P_{t-1}\right)\right] \\
=& E\left[\left(\Delta Q_t\left(\frac{1}{2}\right)S + \epsilon_t\right)\left(\Delta Q_{t-1}\left(\frac{1}{2}\right)S + \epsilon_{t-1}\right)\right] \\
=& E\left[\epsilon_t\epsilon_{t-1} + \epsilon_t\left(\frac{1}{2}\right)S\Delta Q_{t-1} + \epsilon_{t-1}\left(\frac{1}{2}\right)S\Delta Q_t + \left(\frac{1}{4}\right)S^2\Delta Q_t\Delta Q_{t-1}\right] \\
=& \underbrace{E\left[\epsilon_t\epsilon_{t-1}\right]}_{\text{=0, by ind.}} + \left(\frac{1}{2}\right)S\underbrace{E\left[\epsilon_t\Delta Q_{t-1}\right]}_{\text{=0, by ind.}} + \left(\frac{1}{2}\right)S\underbrace{E\left[\epsilon_{t-1}\Delta Q_t\right]}_{\text{=0, by ind.}} + \left(\frac{1}{4}\right)S^2E\left[\Delta Q_t\Delta Q_{t-1}\right] \\
=& \left(\frac{1}{4}\right)S^2E\left[\left(Q_t - Q_{t-1}\right)\left(Q_{t-1} - Q_{t-2}\right)\right] \\
=& \left(\frac{1}{4}\right)S^2E\left[Q_tQ_{t-1} - Q_tQ_{t-2} - Q_{t-1}^2 + Q_{t-1}Q_{t-2}\right] \\
=& \left(\frac{1}{4}\right)S^2\left\{\underbrace{E\left[Q_tQ_{t-1}\right]}_{=0} - \underbrace{E\left[Q_tQ_{t-2}\right]}_{=0} - E\left[Q_{t-1}^2\right] + \underbrace{E\left[Q_{t-1}Q_{t-2}\right]}_{=0}\right\} \\
=& -\left(\frac{1}{4}\right)S^2\underbrace{E\left[Q_{t-1}^2\right]}_{=1} \\
=& -\left(\frac{1}{4}\right)S^2
\end{aligned}
\tag{30}
$$

Rearranging Eq. (30) gives

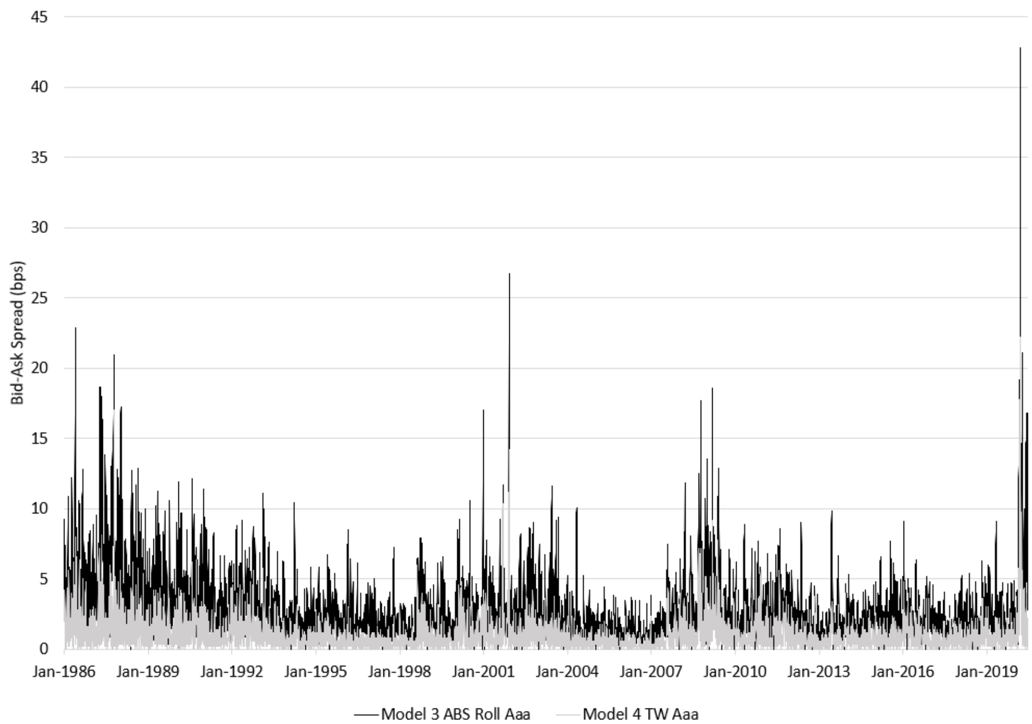
$$
\begin{aligned}
-\left(\frac{1}{4}\right)S^2 =& \mathrm{COV}\left(\Delta P_t, \Delta P_{t-1}\right) \\
S^2 =& -4\mathrm{COV}\left(\Delta P_t, \Delta P_{t-1}\right) \\
S =& \sqrt{-4\mathrm{COV}\left(\Delta P_t, \Delta P_{t-1}\right)} \\
S =& 2\sqrt{-\mathrm{COV}\left(\Delta P_t, \Delta P_{t-1}\right)}
\end{aligned}
\tag{31}
$$

which is the effective bid-ask spread S introduced in Roll (1984).[9] $\square$

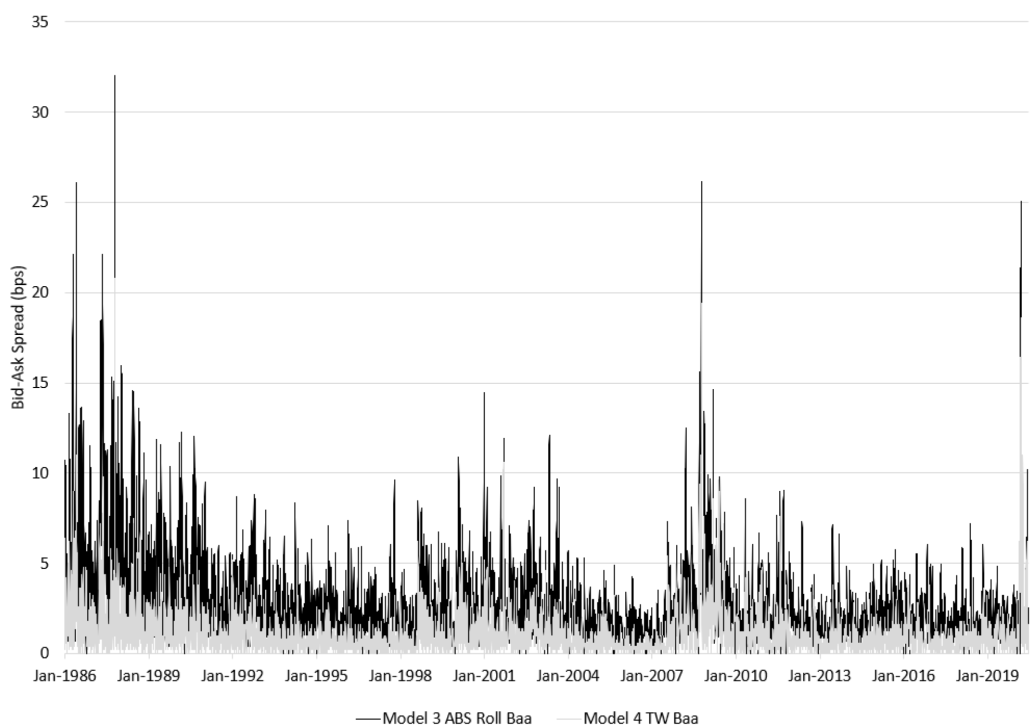
[9] As previously stated in Eq. (1) in the main text.

A.2 Time series of estimated bid-ask spreads

Figure 2: Estimated bid-ask spreads (1986 - 2020, daily)



(a) Estimated Aaa bid-ask spreads



(b) Estimated Baa bid-ask spreads